NONSTATIONARY CONDUCTING FREE FLOW IN A TRANSVERSE MAGNETIC FIELD

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In magnetohydrodynamic flow the viscous friction at the walls can be substantial. The role of viscous friction can be considerably reduced by using a free or a semirestricted flow of the conducting fluid. Nonstationary phenomena in one-dimensional motion of a free plane incompressible fluid flow in a transverse magnetic field are examined. The narrow sides of the flow come into contact with the sectional electrodes connected through external circuits with an active-inductive load. The magnetic Reynolds number and the magnetodydynamic interaction parameter are assumed to be large. When the electric field due to electromagnetic induction in the channel is much smaller than the field due to the external circuits, the problem can be reduced to the characteristic Cauchy problem for a quasilinear hyperbolic system of first-order equations which can be solved by the method of characteristics using a computer.

1. Formulation of the Problem. Let us examine a free plane flow of a nonviscous incompressible conducting fluid moving in a transverse magnetic field **B** $(0, B_y, 0)$ at a velocity **u** $(u_x, 0, 0)$ and with the narrow sides in contact with the electrodes in the planes $z = \pm^1\!/_2 z_0$; in Fig. 1, the free flow is shown by the broken lines between the electrodes. The electrodes are sectioned along the x axis, and each electrode pair is connected to an independent external circuit placed on the downstream side and consisting of an ohmic resistance R_e and an inductance L_e. In the region $y > ^1\!/_2 y_0$, $y < -^1\!/_2 y_0$, there is an ideal magnetic circuit ($\mu = \infty$) closed between the planes $x = 0$ and $x = l$. The flow moves in the narrow gap y_0 between the ferromagnetic walls without touching them. If the external circuits are closed, an electric current with a density **j** $(0, 0, j)$ appears in the stream and magnetohydrodynamic interaction takes place.

It is assumed that at the input of the channel the flow has longitudinal nonconducting walls and at the output of the channel, for instance, the flow begins to break up so that the edge effects can be neglected. The distribution of the external resistances and inductances and also the external magnetic field B_e are considered as independent smooth functions at $x > 0$, $t > 0$: $R_e = R_e(x, t)$, $L_e = L_e(x)$, $B_e = B_e(t)$, $R_e(0, t) \to \infty$ (at the input the electrodes are open).

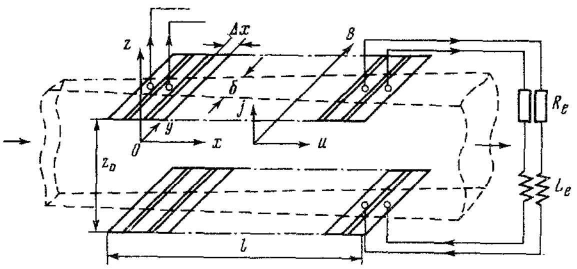

Fig. 1

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The functions R_e and L_e are such that the electric field produced by the external circuits is much larger than the field produced by the change in the internal flow interaction in the channel for the circuit of each electrode pair. The gravitational and bulk dynamic forces act only along the x axis and are characterized by the acceleration $q_x(t)$. The significance of these restrictions will be obvious from the following. The functions $R_e(x, t)$, $L_e(x)$, $B_e(t)$ are given and are the controlling parameters for the system. If necessary, the conductivity of the fluid σ as a function of x and t can be included. However, in this case, $\sigma = \text{const}$. The magnetic Reynolds number R_m and the parameter of the magnetohydrodynamic interaction for the flow are assumed to be considerable.

For $t < 0$ let the flow be steady, and at the instant $t = 0$ let the controlling parameters begin to vary. Let us examine the transient processes in the system.

2. The Initial System of Equations and Its Transformation. One of the most difficult stages in solving one-dimensional nonstationary problems is to obtain information on the electric field in the channel. For $R_m \gg 1$, this field is determined not only by the resistance voltage drop but also by the electromagnetic induction for the external circuits and currents in the channel. The calculation of these effects using Maxwell's equations is difficult since in a one-dimensional approximation the electric field does not necessarily satisfy these equations [1, 2] and apart from that they do not explicitly contain the external circuit parameters. The role of the external inductances as assumed is sufficiently large; therefore it is more convenient to use Kirchhoff's law for the circuit of the transverse flow element of thickness Δx. We have

$$u_x B_y z_0 = I_z\left(\frac{z_0}{\sigma\Delta x\delta} + R_e\right) + L_e\frac{\partial I_z}{\partial t} + \frac{\partial \Psi_i}{\partial t} \tag{2.1}$$

Here I_z is the current of the flow element, δ is the flow thickness along the y axis ($\delta < y_0$), and Ψ_i is the total internal flow interaction of the element in the channel. The quantities R_e, L_e and the relation

$$L^* = \frac{L_e}{R_e + z_0/\sigma\Delta x\delta} \tag{2.2}$$

describing the time scale of the transient process are limited from below in such a way that $\partial\Psi_i/\partial t$ in Eq. (2.1) is not significant.* The eddy fields at the ends of the channel are not taken into account since the end effects are considered to be suppressed.

Taking into account that $I_z = j_z\delta\Delta x$ and $u_x\delta = \text{const}$, (2.1) can be rewritten in the form

$$u_x B_y z_0 = j_z\left(\frac{z_0}{\sigma} + R_e\Delta x\delta\right) + L_e\Delta x\delta\left(\frac{\partial j_z}{\partial t} - \frac{j_z}{u_x}\frac{\partial u_x}{\partial t}\right) \tag{2.3}$$

The other two initial conditions will be given by the equation of motion and the first Maxwell's equation:

$$\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} = -\frac{j_z B_y}{\rho} + q_x, \quad \frac{\partial B_y}{\partial x} = \mu_0 j_z \tag{2.4}$$

where ρ is the fluid density and μ_0 is the permeability of free space.

Let us further relate x to l, t to l/u_0, and let us introduce the dimensionless quantities and parameters

$$u = \frac{u_x}{u_0}, \quad B = \frac{B_y}{B_0}, \quad j = \frac{j_z}{\sigma u_0 B_0}, \quad R = \frac{R_e\Delta x\delta_0\sigma}{z_0}$$
$$L = \frac{L_e\Delta x\delta_0}{\mu_0 z_0 l^2}, \quad q = \frac{q_x l}{u_0^2}, \quad R_m = \mu_0\sigma u_0 l, \quad S = \frac{\sigma B_0^2 l}{\rho u_0}$$

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*In a one-dimensional approximation the restriction on the examined model is connected with the inequality for dimensionless parameters (see below)

$$\frac{\delta_0}{y_0}\left\{(1-x)\int_0^x \frac{\partial}{\partial t}\left[\frac{j(\zeta)}{u(\zeta)}\right]d\zeta + \int_x^1 (1-\zeta)\frac{\partial}{\partial t}\left[\frac{j(\zeta)}{u(\zeta)}\right]d\zeta\right\}\left\{L\frac{\partial}{\partial t}\left[\frac{j(x)}{u(x)}\right]\right\}^{-1} \ll 1,$$

which is satisfied at sufficiently large R_e, L_e, and L^*.

Here R_m is the magnetic Reynolds number and S is the magnetohydrodynamic interaction parameter. Then the initial system will have the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -SjB + q, \qquad \frac{\partial B}{\partial x} = R_m j \tag{2.5}$$

$$\frac{\partial j}{\partial t} + j\frac{\partial u}{\partial x} = \frac{1}{LR_m}[u^2B - j(u+R)] - \frac{Sj^2B}{u} + \frac{qj}{u} \tag{2.6}$$

Equation (2.6) follows from (2.3) and (2.4) after changing to dimensionless quantities.

Let us write the system (2.5), (2.6) in matrix form:

$$A\xi_t + C\xi_x = D \tag{2.7}$$

Here ξ is the unknown vector-function with components u, j, B; A is the matrix obtained from the coefficients in front of the derivatives with respect to t; C is the matrix from the coefficients in front of the derivatives with respect to x; D is the column vector with the components formed by the right-hand sides of the equations (2.5), (2.6). The subscripts denote differentiation with respect to t and x.

It is easy to see that both matrices A and C are singular. This makes the analysis and transformation of the initial system difficult. Therefore let us use new independent variables:

$$\eta = t + x, \qquad \tau = t - x$$

for which we obtain

$$\frac{\partial u}{\partial \tau} + \frac{1+u}{1-u}\frac{\partial u}{\partial \eta} = \frac{1}{1-u}(q - SjB) \tag{2.8}$$

$$\frac{\partial j}{\partial \tau} + \frac{2j}{1-u}\frac{\partial u}{\partial \eta} + \frac{\partial j}{\partial \eta} = \frac{1}{R_m L}[u^2B - j(u+R)] + \frac{j}{u(1-u)}(q - SjB) \tag{2.9}$$

$$\frac{\partial B}{\partial \tau} - \frac{\partial B}{\partial \eta} = -R_m j \tag{2.10}$$

For this system, matrix A is unitary and matrix C is nonsingular. The characteristic directions $\omega = d\eta/d\tau$ of system (2.8)-(2.10) are determined in the usual way [3] after equating the determinant $\| C - \omega I \|$ to zero, where I is unit matrix. We have

$$\omega_1 = (1+u)/(1-u), \qquad \omega_2 = 1, \qquad \omega_3 = -1$$

Since all ω are real and different, the quasilinear system (2.8)-(2.10) is hyperbolic.

For each characteristic curve it is possible to find a left three-dimensional eigenvector λ which satisfies the equation

$$\lambda_i C = \omega_i \lambda_i \qquad (i = 1, 2, 3) \tag{2.11}$$

We have

$$\lambda_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \quad \lambda_2 = \begin{Bmatrix} -j/u \\ 1 \\ 0 \end{Bmatrix}, \quad \lambda_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

Let us multiply Eqs. (2.8)-(2.10) by the components λ_i for each ω_i, and let us sum. Then we will obtain the characteristic normal form [3] of the system where in each equation differentiation is carried out only along the direction of ω_i with the operator $\partial(\cdot)/\partial\tau + \omega_i \partial(\cdot)/\partial\eta$. Since this operator is equivalent to $d(\cdot)/d\tau$ along the corresponding characteristic, we have

$$\frac{du}{d\tau} = \frac{1}{1-u}(q - SjB) \quad \text{along} \quad \left(\frac{d\eta}{d\tau}\right)_1 = \frac{1+u}{1-u}$$

$$\frac{dj}{d\tau} - \frac{j}{u}\frac{du}{d\tau} = \frac{1}{R_m L}[u^2B - j(u+R)] \quad \text{along} \quad \left(\frac{d\eta}{d\tau}\right)_2 = 1$$

$$\frac{dB}{d\tau} = -R_m j \quad \text{along} \quad \left(\frac{d\eta}{d\tau}\right)_3 = -1$$

Using the property of invariance of the characteristic directions with respect to the transformation of coordinates, let us return to the previous independent variables x and t, and we shall finally obtain

$$\frac{du}{dt} = q - SjB \quad \text{along} \quad \left(\frac{dx}{dt}\right)_1 = u \tag{2.12}$$

$$\frac{dj}{dt} - \frac{j}{u}\frac{du}{dt} = \frac{1}{R_m L}[u^2 B - j(u+R)] \quad \text{along} \quad \left(\frac{dx}{dt}\right)_2 = 0 \tag{2.13}$$

$$\frac{dB}{dx} = R_m j \quad \text{along} \quad \left(\frac{dx}{dt}\right)_3 = \infty \tag{2.14}$$

The latter characteristic equation corresponds to the propagation of the electromagnetic excitation at an infinite velocity. Taking into account the relativistic terms, the slope of the characteristic 3 will be determined by the velocity of light [4].

Further, we shall seek only continuous solutions for the unknowns u, j, B, admitting, however, discontinuities in their first derivatives on the characteristics. The investigation of discontinuous solutions of the step type, which are apparently possible for large R_m and S and small L because of the nonlinearity of the initial system, is a separate problem.

The system (2.12)-(2.14) can be obtained directly from the basic condition of indeterminacy of the derivatives on the characteristic curves [5]. Moreover, expressions for total differentials of the unknowns, for instance,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial t}dt, \text{ etc.}$$

are added to the initial system (2.5), (2.6).

By using Cramer's law to solve the system obtained with respect to each derivative and equating the determinants in the numerator and denominator to zero, we obtain an equation for the characteristics (roots of denominator) and compatibility conditions for these (roots of numerator) which coincide with the system (2.12)-(2.14).

The characteristic system (2.12)-(2.14) can be solved numerically if the initial values of u, j, and B on some boundary curve in the xt plane are known. This curve should be three-dimensional or with some additional restrictions of the characteristic type [3, 6] when for any point (x, t) it is possible to find the region which is cut off on the curve by the extreme characteristics passing through the point (x, t) in the direction of the decrease in t. In the examined problem the slopes of all characteristics ω^{-1} are nonnegative, since the perturbations in the flow do not propagate upstream.

For the hydrodynamic perturbations this is explained by the absence of a pressure gradient in the flow, for the current perturbations this is explained by the omission of $\partial\Psi_j/\partial t$ in Eq. (2.1), and for the magnetic field it is explained by the selected geometry of the magnetic circuit and current leads. Actually, in the present model each elementary current j(x') induces a self-magnetic field [2, 7] in the region x > x' only, since, as regards this current, which short-circuits the current leads on the right, part of the nonmagnetic gap in the region x < x' is shunted by the closed external magnetic circuit with $\mu \to \infty$. Thus, the magnetic field at the input remains unperturbed.

3. Initial Conditions. Let us formulate the initial conditions on the x and t axes* with the following assumptions. At the instant t = 0 of the transient process, the current in the flow cannot instantaneously change because of the inductances. Consequently, the functions u(x, 0), j(x, 0), B(x, 0) are known from the preceding steady-state flow conditions. Then the current at the input is equal to zero because of the break of the external circuit, and thus the velocity u(0, t) is determined by the hydrodynamics in the region x < 0. The magnetic field at the input is identically equal to the external field $B_e(t)$. Thus, we have

$$u(x, 0) = \varphi_1(x), \quad j(x, 0) = \varphi_2(x), \quad B(x, 0) = \varphi_3(x) \tag{3.1}$$

$$u(0, t) = \theta_1(t), \; j(0, t) = \theta_2(t) \equiv 0, \quad B(0, t) = \theta_3(t) = B_e(t) \tag{3.2}$$

For the continuity of the solutions it is necessary that matching conditions at the point (0, 0) should be satisfied, i.e.,

$$\varphi_i(0) = \theta_i(0) \quad (i = 1, 2, 3) \tag{3.3}$$

In the examined case the axes x and t will be the characteristics, and therefore we arrive at the characteristic Cauchy problem (Goursat problem). Its special feature is the fact that the functions $\varphi_i(x)$ and $\theta_i(t)$ cannot be given arbitrary values but should satisfy the corresponding characteristic equations. Other-

*On the t axis these conditions can be called boundary conditions.

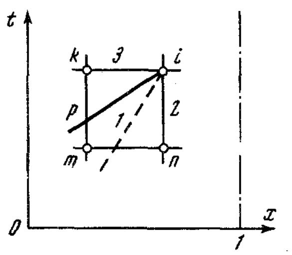

Fig. 2

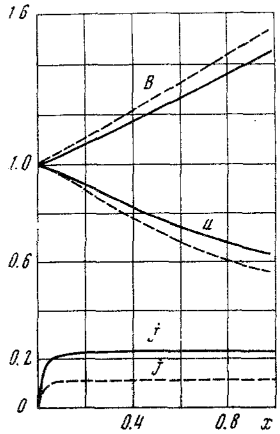

Fig. 3

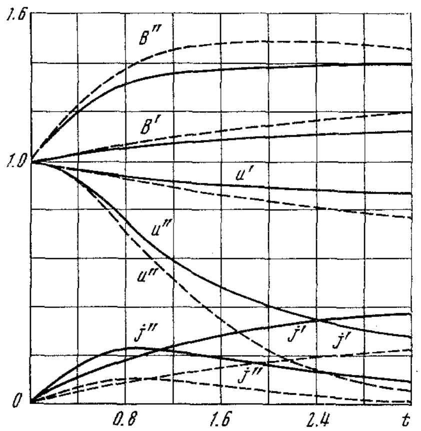

Fig. 4

wise, however, this limitation is expressed by the requirement that the number of conditions on the characteristic boundary should be equal to the number of characteristics included in the region of influence without taking into account the boundary curve itself [3].

4. Numerical Calculations and Discussions of Results. The numerical solution for the system (2.12)-(2.14) was carried out for a quadratic network in the xt plane, a typical cell of which is shown in Fig. 2. For the node i, the region of dependence is determined by the three neighboring nodes k, m, n, which lie in the direction of decrease in x and t. The calculation was made by means of successive displacements from the x and t axes in the region of their influence $0 \leq x \leq t$, $0 \leq t < \infty$. The intervals ik and in lie on characteristics 3 and 2, respectively. The characteristic 1 passed through the point i with a slope corresponding to the intermediate point p located in the region of the function. All parameters at the point p were determined by linear interpolation of the parameters in the neighboring nodes. If the parameters at the points k, m, n are known, it is possible to draw all characteristics at the point i and solve the system (2.12)-(2.14) by the method of finite differences along the characteristic intervals.

The problem with the following initial data was solved:

$$u(x, 0) = u(0, t) = B(x, 0) = B(0, t) = 1, \quad j(x, 0) = j(0, t) = 0 \tag{4.1}$$

It was assumed that $B_0 = B_e = \text{const}$. This type of initial conditions corresponds, for example, to the short-circuiting of the external circuits with a given distribution of resistances and inductances at the instant $t=0$ or switching on at $t=0$ of a constant external magnetic field. In the latter case we neglect the eddy currents induced in the flow at the instant when the field is switched in, or assume their time scale much smaller than the mean ratio L^* determined by (2.2). These initial conditions can also be used with some approximation for the case of input of the flow into the channel at the instant $t=0$ if

$$l / u_0 \ll L^*$$

It can easily be seen that the initial data of (4.1) satisfy the characteristic equations (2.13) and (2.14), which is the necessary condition for the characteristic Cauchy problem. Equation (2.14) for the x axis is identically satisfied, and in Eq. (2.13) for the t axis we have

$$j \equiv 0, \quad \frac{dj}{dt} = \frac{du}{dt} = 0, \quad uB\big|_{x=0} = \underset{\substack{R \to \infty \\ j \to 0}}{j\left(1 + \frac{R}{u}\right)}\Big|_{x=0} = 1$$

The last relation is analogous to Kirchhoff's law for any open source of emf (in our case uB) and is physically obvious.

It should be noted that the restriction assumed above $R(0, t) \to \infty$ is not stringent from a practical point of view. In the general case of an arbitrary distribution of R_e over the electrodes, the origin of coordinates can be slightly displaced upstream, where there are no electrodes and no current, and the function R can be approximated in a one-dimensional approximation with the required features. If the external resistances are distributed uniformly over the electrodes, a good approximation for the calculated function R(x) is given, for example, by the hyperbola $x^{-1/n}$, $n \geq 5$.

The calculations were made by using an IBM-360 (model 91) computer for a network with cell dimensions $\delta x = \delta t = 10^{-2}$, $R = L = x^{-1/5}$, $q = 0$ and different values of R_m and S. Typical results are given in Fig. 3 for t = 1 and also in Fig. 4 for x = 0.19 (the variables are marked by one prime) and for x = 0.79 (the variables are marked by two primes); moreover, the continuous lines show the values $R_m = S = 2$, and the broken lines $R_m = S = 5$. It can be seen that an increase in the conductance σ leading to a proportional increase in R_m and S causes a more drastic change in the parameters in the channel. The apparent decrease in j in this case is connected with an increase in the basis $\sigma u_0 B_0$. However, if R_m and S increase at the expense of l, j decreases because of the redistribution of R_e and L_e, which is necessary for satisfying the accepted condition $R = L = x^{-1/5}$. Moreover, the scales for x and t also vary. From Figs. 3, 4 it follows that at the head of the channel (x = 0.19) j increases monotonically and at the end of the channel (x = 0.79) j increases with t and then decreases because of the considerable decrease in velocity.

The numerical results were spot checked by substituting (2.5), (2.6) in the initial system. The partial derivatives were located approximately from the increase in the quantities in the interval $\delta x' = \delta t' = 0.1$. The error in the equations did not exceed 3% of the maximum term right up to t = 5.

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